Exercises and Discussion Questions
for Bogner Lecture 2-3

1. In microscopic calculations of nuclei and nuclear matter, it is found that the momentum distributions of low-energy states \( n(q) = \langle a_q^\dagger a_q \rangle \) exhibit scaling behavior at large \( q \). In other words, the asymptotic momentum distributions of different nuclei have a universal shape and differ only by a nucleus-dependent scale factor:

\[
\langle \Psi^A_n | n(q) | \Psi^A_n \rangle \sim f(q) C(A, n). \tag{1}
\]

This scaling is often discussed in the context of recent experiments at JLAB that study short-range-correlations (SRCs) in nuclei using double knock-out reactions.

In this exercise, you will construct a low-momentum effective theory and derive this factorization formula for the \( A = 2 \) body system. I claim that this \( A = 2 \) calculation is sufficient to derive the general result in Eq. 1, but that takes a little more space than can be done here!

Note: I use the convention that \( q, q', ... \) denote high momenta and \( p, p', ... \) denote low momenta in what follows.

(a) Consider a two-body system in the center of mass frame so that we reduce the Schrödinger equation to an effective 1-body problem. That is, position and momentum states refer to relative coordinates. In our underlying theory, we have

\[
H | \psi_n^\infty \rangle = E_n | \psi_n^\infty \rangle. \tag{2}
\]

Introduce projection operators \( P + Q = 1, PQ = QP = 0, P^2 = P, Q^2 = Q \) where \( P \) projects onto low-momentum states and \( Q \) onto high-momentum components,

\[
P = \int_0^\Lambda dp |p\rangle\langle p| \quad Q = \int_\Lambda^\infty dq |q\rangle\langle q|
\]

Writing \( H = PHP + PHQ + QHP + QHQ \), use the Schrödinger equation to show the following (Hint: Write eq. 1 in schematic 2x2 form use it to write \( Q | \psi_n \rangle \) in terms of other quantities)

\[
PH_{\text{eff}}(E_n)P | \psi_n^A \rangle = E_n P | \psi_n^A \rangle \quad \text{where} \quad | \psi_n^A \rangle \equiv P | \psi_n^\infty \rangle \tag{4}
\]

where

\[
PH_{\text{eff}}(E_n)P = PHP + PVQ \frac{1}{E_n - QHQ} QVP. \tag{5}
\]
(b) Now consider low-energy states so that $|E_n| \ll \Lambda^2$ and can be neglected. Using your intermediate result in part (a) for $Q\langle \psi_n^\infty \rangle$, derive the following asymptotic formula for $q >> \Lambda$ for the low-energy wave functions of the underlying theory (Hint: You can use $V(q,p) \approx V(q,0)$),

$$
\psi_n^\infty(q) \sim \gamma(q; \Lambda) \int_0^\Lambda dp \psi_n^A(p) = \gamma(q; \Lambda) \psi_n^A(r = 0),
$$

where the $q$-dependence is carried by the state-independent function

$$
\gamma(q; \Lambda) = -\int_\Lambda^\infty dq' \langle q| \frac{1}{QHQ} |q' \rangle V(q',0).
$$

Finally, using (for $A = 2$ system)

$$
\langle \psi_n^\infty | n(q) | \psi_n^\infty \rangle = |\psi_n^\infty(q)|^2 = \gamma^2(q; \Lambda) |\psi_n^A(r = 0)|^2
$$

we have “derived” the factorization formula in equation 1 for the special case of $A = 2$. I claim without proof that using what we have just shown, it is straightforward to derive the general result in Eq. 1 for any $A$-body system.